Factorization of and Determinant Expressions for the Hypersums of Powers of Integers

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Abstract

We derive a compact determinant formula for calculating and factorizing the hypersum polynomials $S_k^{(L)}(N) \equiv \sum_{n_1=1}^{N} \cdots \sum_{n_{L+1}=1}^{n_L} (n_{L+1})^k$ expressed in the variable $N(N+L+1)$.

I. INTRODUCTION

In this article, we will consider the finite sums and hypersums of positive integers raised to a positive integer k. Let $S_k(N)$ denote the sum of such integers from 1 to N:

$$S_k(N) = \sum_{n=1}^{N} n^k. \tag{1}$$

The hypersums $S_k^{(L)}(N)$ are then defined recursively as

$$S_k^{(0)}(N) = S_k(N), \quad S_k^{(L)}(N) = \sum_{n=1}^{N} S_k^{(L-1)}(n) \ \text{ for } \ L \geq 1. \tag{2}$$

In particular,

$$S_0^{(1)}(N) = \sum_{n=1}^{N} S_0^{(0)}(n) = \sum_{n=1}^{N} n = S_1^{(0)}(N), \tag{3}$$

from which it follows that $S_0^{(L)}(N) = S_1^{(L-1)}(N)$ for all $L \geq 1$. In the following, we will mostly take $k > 0$. $S_k^{(L)}(N)$ is an $(L+k+1)$-order polynomial in N, and is given by the formula [1]

$$S_k^{(L)}(N) = \sum_{q=0}^{k} S(k,q)\ q! \begin{pmatrix} N+L+1 \\ q+L+1 \end{pmatrix} \tag{4}$$

where $S(k,q)$ is a Stirling number of the 2nd kind. As an example, we quote from Knuth [1] a partial result of Faulhaber for $S_6^{(10)}(N)$ (which in Knuth's notation is $\Sigma^{11} n^6$):

$$S_6^{(10)}(N) = \frac{5!}{17!}\ \left\{\ 6\ N^{17} + 561\ N^{16} + \cdots + 1021675563656\ N^5 + \cdots - 96598656000\ N\ \right\}.$$

For $N = -1, -2, \ldots, -L-1$, the sum over q on the right side of eq.(4) is zero since the binomial coefficients are all zero; the sum is also zero for $N = 0$ if $k > 0$, since in this case $S(k,0) = 0$ and all $q > 0$ binomial coefficients are zero. The polynomial representing $S_k^{(L)}(N)$ for $k > 0$ therefore has zeros at these values, and is expressible in the form

$$S_k^{(L)}(N) = N(N+1)(N+2)\cdots(N+L+1) \times \mathcal{Q}^{(k-1)}(N) \tag{5}$$

where $\mathcal{Q}^{(k-1)}(N)$ is a $(k-1)$-order polynomial in N.

While the $S_k^{(L)}$ polynomial can be calculated using eq.(4), this formula does not readily lend itself to expression in the factored form of eq.(5). In addition, it is known that these polynomials simplify when expressed in the variable $N(N+L+1)$. The aim of this article

is to derive an alternative formula for the $S_k^{(L)}$ polynomial in this variable, one in which the zeros at $N = 0, -1, \ldots, -L-1$ are factored out and in which the remaining $(k-1)$-order polynomial is given as a determinant of a relatively simple matrix, one that does not explicitly involve Stirling numbers (or any other "complicated" numbers). We do this in Theorem 3 in Section IV; in particular, our result for $S_6^{(10)}$ is

$$S_6^{(10)}(y) = \frac{6!}{17!}\,\frac{\sqrt{4y+121}}{2}\;y(y+10)(y+18)(y+24)(y+28)(y+30) \times \frac{3y^2+22y-220}{3},$$

where $y \equiv N(N+11)$. To do this, we need to develop some machinery, so for now we will consider only the $L = 0$ polynomials. These are given by Faulhaber's formula [2]:

$$S_k(N) = \frac{1}{k+1}\sum_{n=1}^{k+1}(-1)^{\delta_{nk}}\begin{pmatrix} k+1 \\ n \end{pmatrix} B_{k+1-n}N^n, \tag{6}$$

where B_n is a Bernoulli number. In the following section, we will derive an alternative expression for S_k which seems to be more convenient in extending the formalism to hypersums.

II. SERIES EXPANSION

In the sum over n for $S_{k+1}(N)$ we make the substitution $n \to m+1$, add and subtract terms to make the sum on m go from 1 to N, and then expand $(m+1)^{k+1}$ binomially, with the result [2, 3]:

$$(k+1)S_k(N) = (N+1)^{k+1} - 1 - \sum_{q=0}^{k-1}\begin{pmatrix} k+1 \\ q \end{pmatrix} S_q(N). \tag{7}$$

Repeating this procedure but now with the replacement $n \to m-1$, we get:

$$(k+1)S_k(N) = N^{k+1} - \sum_{q=0}^{k-1}\begin{pmatrix} k+1 \\ q \end{pmatrix}(-1)^{k-q}S_q(N). \tag{8}$$

Adding (7) and (8) and dividing by $2(k+1)$, the odd-q terms in the sum cancel if k is an even integer, while for odd k the even-q terms cancel. We therefore get separate recursion

relations for the even and odd power sums:

$$S_{2p}(N) = \frac{1}{2p+1}\left[\frac{(N+1)^{2p+1}+N^{2p+1}-1}{2} - \sum_{q=0}^{p-1}\begin{pmatrix}2p+1\\2q\end{pmatrix}S_{2q}(N)\right], \tag{9a}$$

$$S_{2p+1}(N) = \frac{1}{2p+2}\left[\frac{(N+1)^{2p+2}+N^{2p+2}-1}{2} - \sum_{q=0}^{p-1}\begin{pmatrix}2p+2\\2q+1\end{pmatrix}S_{2q+1}(N)\right]. \tag{9b}$$

We now define x as $2N+1$ and the functions $f_p(x)$ and $g_p(x)$ as

$$f_p(x) = \frac{1}{2^{2p+2}}\left[(x+1)^{2p+1}+(x-1)^{2p+1}\right]-\frac{x}{2}, \tag{10a}$$

$$g_p(x) = \frac{1}{2^{2p+3}}\left[(x+1)^{2p+2}+(x-1)^{2p+2}\right]-\frac{1}{2}-(p+1)\,\frac{x^2-1}{4}. \tag{10b}$$

We will assume for now that $p>0$. Equations (9a) and (9b) then become

$$S_{2p}(x) = \frac{1}{2p+1}\left[f_p(x) - \sum_{q=1}^{p-1}\begin{pmatrix}2p+1\\2q\end{pmatrix}S_{2q}(x)\right], \tag{11a}$$

$$S_{2p+1}(x) = \frac{1}{2p+2}\left[g_p(x) - \sum_{q=1}^{p-1}\begin{pmatrix}2p+2\\2q+1\end{pmatrix}S_{2q+1}(x)\right]. \tag{11b}$$

(Note that we have absorbed the $q=0$ terms in the sums in (9a) and (9b) into f_p and g_p, respectively.)

$f_p(x)$ is an odd function of x, while $g_p(x)$ is even. Further, $f_p(x)$ has zeros at $x=0,\pm1$, and $g_p(x)$ has double zeros at $x=\pm1$. $S_{2p}(x)$ and $S_{2p+1}(x)$ are therefore odd and even functions of x, respectively, with zeros at these points. These sums can therefore be expanded in these functions, which we do as

$$S_{2p} = (2p)!\sum_{q=1}^{p}\frac{C^{(1)}_{p-q}}{(2q+1)!}\;f_q, \tag{12a}$$

$$S_{2p+1} = (2p+1)!\sum_{q=1}^{p}\frac{C^{(2)}_{p-q}}{(2q+2)!}\;g_q. \tag{12b}$$

Equations (11) fix the values $C^{(1)}_0 = C^{(2)}_0 = 1$. We now substitute (12a) into (11a) and (12b) into (11b) to get

$$\sum_{q=0}^{p}\frac{C^{(1)}_{p-q}}{(2q+1)!} = \sum_{q=0}^{p}\frac{C^{(2)}_{p-q}}{(2q+1)!} = 0 \qquad (p>0). \tag{13a}$$

These two sets of coefficients satisfy the same recursion relation and have the same initializing value; they are therefore equal, and we will denote the pth coefficient simply as C_p. These coefficients satisfy the additional recursion relations:

$$\sum_{q=0}^{p} \frac{2^{2q} C_{p-q}}{(2q+1)!} = \frac{1}{(2p)!}, \tag{13b}$$

$$\sum_{q=0}^{p} \frac{2^{2q+1} C_{p-q}}{(2q+2)!} = \frac{1}{(2p+1)!}, \tag{13c}$$

$$\sum_{q=0}^{p} \frac{2^{2p-2q} C_{p-q}}{(2q)!} = C_p, \tag{13d}$$

$$\sum_{q=0}^{p} \frac{2^{2p-2q} C_{p-q}}{(2q+1)!} = \frac{E_{2p}}{(2p)!}, \tag{13e}$$

$$\sum_{q=0}^{p} \frac{E_{2q} C_{p-q}}{(2q)!} = 2^{2p} C_p, \tag{13f}$$

$$\sum_{q=0}^{p} \frac{C_{p-q}}{(2q)!} = \frac{C_p}{2^{1-2p}-1}, \tag{13g}$$

$$\sum_{q=0}^{p} \frac{1}{(2q+1)!} \frac{C_{p-q}}{2^{2p-2q}-2} = -\frac{\delta_{0p}}{2} - \frac{1}{2(2p)!}, \tag{13h}$$

$$\sum_{q=0}^{p} \frac{1}{(2q)!} \frac{C_{p-q}}{2^{2p-2q}-2} = \frac{C_p}{2^{2p}-2} - \frac{1}{2(2p-1)!}, \tag{13i}$$

$$\sum_{q=0}^{p} C_q C_{p-q} = \frac{2p-1}{1-2^{1-2p}} \, C_p, \tag{13j}$$

and in general, for N an integer > 1,

$$\sum_{q=0}^{p} \frac{C_{p-q}}{(2q+1)!} \, N^{2q+1} = \frac{2}{(2p)!} \left[(N-1)^{2p} + (N-3)^{2p} + \cdots + \left\{ \begin{array}{c} 1 \\ 2^{2p} \end{array} \right. \right] \left\{ \begin{array}{l} N \text{ even} \\ N \text{ odd } (p > 0) \end{array} \right\}, \tag{13k}$$

$$\begin{aligned} \sum_{q=0}^{p} \frac{C_{p-q}}{(2q)!} \, N^{2q} = {} & \left\{ \begin{array}{c} 1 \\ (2^{1-2p}-1)^{-1} \end{array} \right\} C_p \\ & + \frac{2}{(2p-1)!} \left[(N-1)^{2p-1} + (N-3)^{2p-1} + \cdots + \left\{ \begin{array}{c} 1 \\ 2^{2p-1} \end{array} \right. \right] \left\{ \begin{array}{l} N \text{ even} \\ N \text{ odd} \end{array} \right\}. \end{aligned} \tag{13l}$$

In (13e) and (13f), E_k is the kth Euler number. For the derivation of relations (13b-13l), see Appendix A.

Consider now the infinite sum $\sum_{p=0}^{\infty} C_p x^{2p}$, which we assume to be convergent in some region. From (13a) we have

$$\begin{aligned}
\sum_{p=0}^{\infty} C_p x^{2p} &= 1 - \sum_{p=1}^{\infty} \sum_{q=0}^{p-1} \frac{C_q}{(2p-2q+1)!}\, x^{2p} \\
&= 1 - \sum_{q=0}^{\infty} C_q \sum_{p=q+1}^{\infty} \frac{x^{2p}}{(2p-2q+1)!} \\
&= 1 - \sum_{q=0}^{\infty} C_q \sum_{n=1}^{\infty} \frac{x^{2q+2n}}{(2n+1)!} \\
&= 1 - \sum_{q=0}^{\infty} C_q x^{2q} \left(\frac{\sinh x}{x} - 1 \right), \qquad (14)
\end{aligned}$$

from which we get

$$\sum_{p=0}^{\infty} C_p x^{2p} = \frac{x}{\sinh x} \qquad (15)$$

As a consequence, the C_p coefficients are related to the Bernoulli numbers as

$$C_p = \frac{2 - 2^{2p}}{(2p)!}\, B_{2p}, \qquad (16)$$

and correspond to integer sequences A036280 (numerators) and A036281 (denominators) in the Encyclopedia of Integer Sequences [4]. The Bernoulli numbers are related to the Riemann zeta function $\zeta(s)$ for s an even, positive integer ([8], p. 12):

$$B_{2p} = (-1)^{p+1} \frac{2(2p)!}{(2\pi)^{2p}}\, \zeta(2p). \qquad (17)$$

Combining this with (16),

$$C_p = (-1)^p \frac{2}{\pi^{2p}} (1 - 2^{1-2p}) \zeta(2p). \qquad (18)$$

But since

$$\left(1 - \frac{2}{2^x}\right) \zeta(x) = \left(1 - \frac{2}{2^x}\right) \sum_{n=1}^{\infty} \frac{1}{n^x} = 1 - \frac{1}{2^x} + \frac{1}{3^x} - \frac{1}{4^x} + \frac{1}{5^x} - \cdots \qquad (19)$$

we have

$$C_p = (-1)^p \frac{2}{\pi^{2p}} \left[1 - \frac{1}{2^{2p}} + \frac{1}{3^{2p}} - \frac{1}{4^{2p}} + \frac{1}{5^{2p}} - \cdots \right] = (-1)^p \frac{2}{\pi^{2p}}\, \eta(2p), \qquad (20)$$

where $\eta(x)$ is the Dirichlet eta function.

Recursion relation (13a) corresponds to eq.(3.2) in Van Malderen [6], and these C_p coefficients are, up to a factor of $(-1)^p$, the same as Van Malderen's D_p coefficients in that article; (see also the article by Chen [7]). Relations (13g) and (13l), for $N = 3$, correspond to eqs. (2.2) and (2.3), respectively, in Van Malderen.

Theorem 1 *For $k \geq 0$, $S_k(x)$ is given by the expression*

$$S_k(x) = \frac{k!}{2^{k+1}} \sum_{q=0}^{\lfloor k/2 \rfloor} C_q \, \frac{x^{k+1-2q} - 1}{(k+1-2q)!}.$$

Proof. The sums in eqs.(12) for S_{2p} and S_{2p+1} can be extended down to $q = 0$ since $f_0 = g_0 = 0$. The resulting sums are however not valid for the $p = 0$ sums S_0 and S_1. They can be made so by adding and subtracting terms inside the summation which sum to zero for $p > 0$ as a consequence of recursion relation (13a). The "corrected" sums are:

$$S_{2p}(x) \;=\; (2p)! \sum_{q=0}^{p} \frac{C_{p-q}}{(2q+1)!} \left\{ \frac{(x+1)^{2q+1} + (x-1)^{2q+1}}{2^{2q+2}} - \frac{1}{2} \right\}, \tag{21a}$$

$$S_{2p+1}(x) \;=\; (2p+1)! \sum_{q=0}^{p} \frac{C_{p-q}}{(2q+2)!} \left\{ \frac{(x+1)^{2q+2} + (x-1)^{2q+2}}{2^{2q+3}} - \frac{1}{2} \right\}. \tag{21b}$$

Expanding the functions inside the brackets in powers of x, we have:

$$\begin{aligned} \frac{(x+1)^{2q+1} + (x-1)^{2q+1}}{2^{2q+2}} - \frac{1}{2} &= \frac{1}{2^{2q+1}} \sum_{n=0}^{q} \binom{2q+1}{2n+1} x^{2n+1} - \frac{1}{2} \\ &= \frac{1}{2^{2q+1}} \sum_{n=0}^{q} \binom{2q+1}{2n+1} (x^{2n+1} - 1), \end{aligned} \tag{22}$$

and

$$\frac{(x+1)^{2q+2} + (x-1)^{2q+2}}{2^{2q+3}} - \frac{1}{2} = \frac{1}{2^{2q+2}} \sum_{n=0}^{q} \binom{2q+2}{2n+2} (x^{2n+2} - 1). \tag{23}$$

Then,

$$\begin{aligned} S_{2p}(x) &= (2p)! \sum_{q=0}^{p} \frac{C_{p-q}}{(2q+1)!} \, \frac{1}{2^{2q+1}} \sum_{n=0}^{q} \binom{2q+1}{2n+1} (x^{2n+1} - 1) \\ &= (2p)! \sum_{n=0}^{p} \frac{x^{2n+1} - 1}{(2n+1)!} \sum_{q=n}^{p} \frac{2^{-2q-1} C_{p-q}}{(2q-2n)!}. \end{aligned} \tag{24}$$

The sum over q is evaluated using (13d):

$$\sum_{q=n}^{p} \frac{2^{-2q-1}C_{p-q}}{(2q-2n)!} = \sum_{l=0}^{p-n} \frac{2^{-2n-1-2l}C_{p-n-l}}{(2l)!} = \frac{C_{p-n}}{2^{2p+1}}, \tag{25}$$

and so we have

$$S_{2p}(x) = \frac{(2p)!}{2^{2p+1}} \sum_{n=0}^{p} C_{p-n} \, \frac{x^{2n+1}-1}{(2n+1)!} = \frac{(2p)!}{2^{2p+1}} \sum_{q=0}^{p} C_q \, \frac{x^{2p+1-2q}-1}{(2p+1-2q)!}. \tag{26}$$

Similarly, for S_{2p+1}:

$$\begin{aligned} S_{2p+1}(x) &= (2p+1)! \sum_{q=0}^{p} \frac{C_{p-q}}{(2q+2)!} \, \frac{1}{2^{2q+2}} \sum_{n=0}^{q} \begin{pmatrix} 2q+2 \\ 2n+2 \end{pmatrix} (x^{2n+2}-1) \\ &= \frac{(2p+1)!}{2} \sum_{n=0}^{p} \frac{x^{2n+2}-1}{(2n+2)!} \sum_{q=n}^{p} \frac{2^{-2q-1}C_{p-q}}{(2q-2n)!}. \end{aligned} \tag{27}$$

The sum over q is the same as in (24) above, and so we get

$$S_{2p+1}(x) = \frac{(2p+1)!}{2^{2p+2}} \sum_{n=0}^{p} C_{p-n} \frac{x^{2n+2}-1}{(2n+2)!} = \frac{(2p+1)!}{2^{2p+2}} \sum_{q=0}^{p} C_q \, \frac{x^{2p+2-2q}-1}{(2p+2-2q)!}. \tag{28}$$

QED

It is interesting to compare the expression in Theorem 1 with Faulhaber's formula. Converting to Bernoulli numbers and expressed in the variable N, it becomes.

$$S_k(N) = \frac{1}{2^k(k+1)} \sum_{n=1}^{k+1} \begin{pmatrix} k+1 \\ n \end{pmatrix} (1-2^{k-n}) B_{k+1-n} [(2N+1)^n - 1], \tag{29}$$

The power-sum polynomials take a simpler form in the variable $y = N(N+1)$ compared to x. (The fact that the polynomials for the odd-power sums simplified when expressed in thid variable was known by Faulhaber; see the discussion by Knuth [1].) In this variable, the polynomials are:

$$\begin{aligned} S_0 &= \frac{\sqrt{4y+1}}{2} - \frac{1}{2}, & S_1 &= \frac{y}{2}, \\ S_2 &= \frac{\sqrt{4y+1}\ y}{6}, & S_3 &= \frac{y^2}{4}, \\ S_4 &= \frac{\sqrt{4y+1}\ y(3y-1)}{30}, & S_5 &= \frac{y^2(2y-1)}{12}, \\ S_6 &= \frac{\sqrt{4y+1}\ y(3y^2-3y+1)}{52}, & S_7 &= \frac{y^2(3y^2-4y+2)}{24}, \\ S_8 &= \frac{\sqrt{4y+1}\ y(5y^3-10y^2+9y-3)}{90}, & S_9 &= \frac{y^2(y-1)(2y^2-3y+3)}{20}, \\ &\vdots & &\vdots \end{aligned} \tag{30}$$

Corollary 1 *The power sum of odd integers,*

$\bar{S}_k(N) \equiv 1 + 3^k + 5^k + \cdots + N^k, \quad N = \text{odd integer}.$

is given by the expression,

$$\bar{S}_k(N) = \frac{k!}{2} \sum_{q=0}^{\lfloor k/2 \rfloor} C_q \, \frac{(N+1)^{k+1-2q}}{(k+1-2q)!}.$$

Proof : This follows directly from relations (13k) and (13l).

Therefore, expressed in the variable

$$\bar{y} = \frac{N(N+2)}{4}, \tag{31}$$

the $\bar{S}_k$ are, up to an overall constant and a factor of 2^k, given by the same polynomials as in (30):

$$\bar{S}_k(\bar{y}) = 2^k \left[S_k(\bar{y}) - S_k\left(-\frac{1}{4}\right) \right]. \tag{32}$$

III. MATRIX FORMALISM

Recursion relation (13a) can be written in triangular-matrix form as

$$\begin{pmatrix} 1 & & & \\ \frac{1}{3!} & 1 & \text{\Large 0} & \\ \frac{1}{5!} & \frac{1}{3!} & 1 & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{33}$$

Lower-triangular matrices which have 1's along their main diagonal are unit lower-triangular matrices. Matrices which are constant along all diagonals are Toeplitz matrices. The matrix above is thus a unit lower-triangular Toeplitz (LTT) matrix.

If A is an $n \times n$ lower-triangular matrix, $(n \leq \infty)$, then, for $k < n$, the $k-$*truncation* of A is the $k \times k$ lower-triangular matrix obtained from A by removing all rows and columns greater than k. It is straightforward to show that the product of two truncated lower-triangular matrices is equal to the truncation of the product and, consequently, the inverse of a truncated lower-triangular matrix is the truncation of the inverse matrix.

We define the infinite-dimensional lower shift matrix J as

$$J \equiv \begin{pmatrix} 0 & & & \\ 1 & 0 & \text{\huge 0} & \\ 0 & 1 & 0 & \\ \vdots & & & \ddots \end{pmatrix}; \qquad (J^p)_{ij} = \delta_{p,i-j}; \qquad J^p J^q = J^{p+q}. \tag{34}$$

and, for future reference, the unit column vectors $\mathbf{I}_p$

$$\mathbf{I}_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}; \qquad \mathbf{I}_p \equiv J^{p-1}\mathbf{I}_1. \tag{35}$$

Any infinite-dimensional LTT matrix A can be expanded out in powers of J, with J^0 being the identity matrix:

$$A = \begin{pmatrix} a_0 & & & \\ a_1 & a_0 & \text{\huge 0} & \\ a_2 & a_1 & a_0 & \\ \vdots & & & \ddots \end{pmatrix} = \sum_{q=0}^{\infty} a_q J^q. \tag{36}$$

LTT matrices commute with one another since J^p commutes with J^q. The determinant of a finite-dimensional lower-triangular matrix is the product of its diagonal elements, so all finite unit lower-triangular matrices have determinant 1.

We will be concerned in the following with determinants that have the general structure:

$$\begin{vmatrix} a_0 & & & & c_0 \\ a_1 & a_0 & \text{\huge 0} & & c_1 \\ \vdots & & \ddots & & \vdots \\ a_{k-2} & & & a_0 & c_{k-2} \\ a_{k-1} & a_{k-2} & \cdots & a_1 & c_{k-1} \end{vmatrix} \tag{37}$$

To describe them, we introduce some notation: The determinant above is of a $k \times k$ matrix consisting of the sum of a "almost LTT" matrix, one whose (k,k) element is zero, (the 'base' matrix), and a matrix whose only non-zero elements are along the last column, (the 'tower'

matrix). For $0 \leq q \leq k-1$, we define the "unit tower matrix" K_q to be a $k \times k$ matrix whose elements are all zero except for the $(k-q, k)$ element, whose value is 1:

$$K_q \equiv \left(\begin{array}{c|c} & 0 \\ & \vdots \\ 0_{(k,k-1)} & 1 \\ & \vdots \\ & 0 \end{array} \right) \begin{array}{l} \\ \\ \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} q \end{array} . \tag{38}$$

(The dimension k of these matrices will usually be clear from the context.) Matrices such as A in (36) and others defined below are infinite-dimensional, whereas in taking determinants the matrices are finite-dimensional. To keep the notation simple, when taking determinants we will use the same symbols for the truncated, finite-dimensional matrices as the infinite-dimensional ones, but will indicate with a subscript the dimensionality of the matrices inside the brackets. In this notation, the determinant in (37) is

$$\det \left\{ A - a_0 K_0 + \sum_{q=0}^{k-1} c_{k-1-q} K_q \right\}_{(k)} . \tag{39}$$

We can also use this notation when A is a non-Toeplitz, lower-triangular matrix with constant diagonal element a_0.

Some useful identities are given in the following lemma:

Lemma 1 *Let A be a lower-triangular matrix which is constant ($= a_0$) along the main diagonal, T be an arbitrary tower matrix, L be a unit LTT matrix, and D be a diagonal matrix: $D = \mathrm{diag}\ \{d_1, d_2, \ldots\}$. Then*

$$\begin{array}{lrcl} \text{(I)} & \det\{A - a_0 K_0 + xT\}_{(k)} & = & x\ \det\{A - a_0 K_0 + T\}_{(k)}, \\ \text{(II)} & \det \left\{ A - a_0 K_0 + \sum_{q=0}^{k-1} c_{k-1-q} K_q \right\}_{(k)} & = & \sum_{q=0}^{k-1} c_{k-1-q}\ \det\left\{A - a_0 K_0 + K_q\right\}_{(k)}, \\ \text{(III)} & \det\{A - a_0 K_0 + LT\}_{(k)} & = & \det\{L^{-1} A - a_0 K_0 + T\}_{(k)} \\ \text{(IV)} & \det\{A - a_0 K_0 + DT\}_{(k)} & = & d_k \det\{D^{-1} A D - a_0 K_0 + T\}_{(k)} \end{array}$$

Proof: The first identity is a statement in this notation of the well-know property that a common factor of any column in a matrix can be factored out of its determinant. The 2nd

identity corresponds to an expansion by minors of the determinant on the left along the kth column. The 3rd and 4th identities follow from the identity $(\det X)(\det Y) = \det(XY)$, and the easily demonstrated relations ($k \times k$ truncation assumed): $L^{-1}K_0 = K_0$, $D^{-1}K_0 D = K_0$, and $TD = d_k T$.

In particular, for tower matrices T_1 and T_2, we have from (II),

$$\det\{A - a_0 K_0 + T_1 + T_2\}_{(k)} = \det\{A - a_0 K_0 + T_1\}_{(k)} + \det\{A - a_0 K_0 + T_2\}_{(k)} . \tag{40}$$

We define the matrices P and Q as

$$P \equiv \begin{pmatrix} 1 & & & \\ \frac{1}{3!} & 1 & \text{\Large 0} & \\ \frac{1}{5!} & \frac{1}{3!} & 1 & \\ \vdots & & & \ddots \end{pmatrix} = \sum_{q=0}^{\infty} \frac{J^q}{(2q+1)!}; \tag{41a}$$

$$Q \equiv \begin{pmatrix} 1 & & & \\ \frac{1}{2!} & 1 & \text{\Large 0} & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & \\ \vdots & & & \ddots \end{pmatrix} = \sum_{q=0}^{\infty} \frac{J^q}{(2q)!}. \tag{41b}$$

Then the C_p coefficients are

$$C_p = \det\{P - K_0 + K_p\}_{(k)} \tag{42}$$

for any $k > p$. This is demonstrated by noting that $C_0 = 1$ by this formula and that

$$\begin{aligned} \sum_{q=0}^{p} \frac{C_{p-q}}{(2q+1)!} &= \sum_{q=0}^{p} \frac{1}{(2q+1)!} \det\{P - K_0 + K_{p-q}\}_{(k)} \\ &= \det\left\{P - K_0 + \sum_{q=0}^{p} \frac{J^q}{(2q+1)!} K_p\right\}_{(k)} \end{aligned} \tag{43}$$

is zero since the last column and the $(p+1)$ column in the determinant are equal. It follows from (42) that

$$P^{-1} = \sum_{q=0}^{\infty} C_q J^q \tag{44}$$

Written out, and expanding by minors, the RHS of eq. (42) is the $p \times p$ determinant

$$C_p = (-1)^p \begin{vmatrix} \frac{1}{3!} & 1 & 0 & \cdots & 0 \\ \frac{1}{5!} & \frac{1}{3!} & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ \frac{1}{(2p-1)!} & \frac{1}{(2p-3)!} & \frac{1}{(2p-5)!} & \cdots & 1 \\ \frac{1}{(2p+1)!} & \frac{1}{(2p-1)!} & \frac{1}{(2p-3)!} & \cdots & \frac{1}{3!} \end{vmatrix} \tag{45}$$

which is the form, up to the factor $(-1)^p$, that Van Malderen gives for his D_p coefficients [6].

We now apply this formalism to the power sums $S_k(x)$ for $k \geq 2$:

$$S_{2p}(x) \;=\; \frac{(2p)!}{2^{2p+1}} \sum_{n=0}^{p} C_{p-n}\, \frac{x^{2n+1}}{(2n+1)!}; \quad S_{2p+1}(x) = \frac{(2p+1)!}{2^{2p+2}} \sum_{n=0}^{p} C_{p-n} \frac{x^{2n+2}-1}{(2n+2)!}. \tag{46}$$

We now define the diagonal matrices D_m as

$$D_m \equiv \begin{pmatrix} m! & & & 0 \\ & (m+2)! & & \\ & & (m+4)! & \\ 0 & & & \ddots \end{pmatrix}, \tag{47}$$

where m takes on nonnegative integer values. These matrices satisfy the identities

$$D_m^{\pm 1} J^p \mathbf{I}_1 \;=\; [(m+2p)!]^{\pm 1}\; J^p \mathbf{I}_1, \tag{48}$$

as well as the corresponding identities for the truncated matrices, with $\mathbf{I}_1$ replaced by an appropriate unit tower matrix.

We have

$$\begin{aligned} S_{2p}(x) \;&=\; \frac{(2p)!}{2^{2p+1}} \sum_{n=0}^{p} \frac{x^{2n+1}}{(2n+1)!}\; \det\{P - K_0 + J^n K_p\}_{(p+1)} \\ &= \frac{(2p)!}{2^{2p+1}}\; \det\left\{P - K_0 + \sum_{n=0}^{p} \frac{x^{2n+1}}{(2n+1)!}\; J^n K_p\right\}_{(p+1)} \\ &= \frac{(2p)!\; x}{2^{2p+1}}\; \det\left\{P - K_0 + D_1^{-1} \sum_{n=0}^{p} x^{2n} J^n K_p\right\}_{(p+1)} \\ &= \frac{(2p)!\; x}{2^{2p+1}}\; \det\left\{P - K_0 + D_1^{-1}\; \frac{1}{I - x^2 J}\; K_p\right\}_{(p+1)}. \end{aligned} \tag{49a}$$

And, in a like fashion,

$$
\begin{aligned}
S_{2p+1}(x) &= \frac{(2p+1)!}{2^{2p+2}} \sum_{n=0}^{p} \frac{x^{2n+2}-1}{(2n+2)!} \det\{P - K_0 + J^n K_p\}_{(p+1)} \\
&= \frac{(2p+1)!(x^2-1)}{2^{2p+2}} \det\left\{P - K_0 + D_2^{-1} \frac{1}{(I-J)(I-x^2J)} K_p\right\}_{(p+1)} .
\end{aligned} \tag{49b}
$$

Note that

$$
(I-J)D_1PD_1^{-1}\mathbf{I}_1 = (I-J)^2D_2PD_2^{-1}\mathbf{I}_1 = \mathbf{I}_1 \tag{50}
$$

I.e., the first column of $(I-J)D_1PD_1^{-1}$ and of $(I-J)^2D_2PD_2^{-1}$ consists of 0's except for the $(1,1)$ element, which is 1. As a consequence,

$$
\det\left\{P - K_0 + D_1^{-1} \frac{1}{I-J} K_p\right\}_{(p+1)} = \det\left\{(I-J)D_1PD_1^{-1} - K_0 + K_p\right\}_{(p+1)} = 0, \tag{51a}
$$

$$
\det\left\{P - K_0 + D_2^{-1} \frac{1}{(I-J)^2} K_p\right\}_{(p+1)} = \det\left\{(I-J)^2D_2PD_2^{-1} - K_0 + K_p\right\}_{(p+1)} = 0. \tag{51b}
$$

Using these identities, we can thus subtract off the tower matrices $D_1^{-1}(I-J)^{-1}K_p$ and $D_2^{-1}(I-J)^{-2}K_p$ from the tower matrices in (49a) and (49b), respectively, without changing the values of the determinants, to get,

$$
S_{2p}(x) = \frac{(2p)!\; x(x^2-1)}{2^{2p+1}} \det\left\{P - K_0 + D_1^{-1} \frac{J}{(I-J)(I-x^2J)} K_p\right\}_{(p+1)} ; \tag{52a}
$$

$$
S_{2p+1}(x) = \frac{(2p+1)!\; (x^2-1)^2}{2^{2p+2}} \det\left\{P - K_0 + D_2^{-1} \frac{J}{(I-J)^2(I-x^2J)} K_p\right\}_{(p+1)} . \tag{52b}
$$

The matrix J, acting on K_p inside the brackets, lowers it to K_{p-1}. The $(1,p+1)$ element of the matrix in each determinant is therefore zero, and we can expand each by minors along the first row, reducing them to $p \times p$ determinants. Changing to the variable y and using identities (III) and (IV) in Lemma 1, we get:

$$
S_{2p}(x) = \frac{y\,\sqrt{4y+1}}{2^{2p-1}(2p+1)} \det\left\{[\, I-(4y+1)J\,](I-J)D_3PD_3^{-1} - K_0 + K_{p-1}\right\}_{(p)} ; \tag{53a}
$$

$$
S_{2p+1}(x) = \frac{y^2}{2^{2p-2}(2p+2)} \det\left\{[\, I-(4y+1)J\,](I-J)^2D_4PD_4^{-1} - K_0 + K_{p-1}\right\}_{(p)} . \tag{53b}
$$

Theorem 3 in the next section generalizes these expressions to the $L > 0$ cases.

IV. HYPERSUMS

We generalize relation (44) and define the coefficients $C_p^{(L)},\ L \geq 0$, by

$$P^{-(L+1)} = \sum_{p=0}^{\infty} C_p^{(L)} J^p. \tag{54}$$

We then have

$$C_p^{(0)} = C_p, \quad C_0^{(L)} = 1, \quad C_p^{(L)} = \sum_{q=0}^{p} C_q^{(L-1)} C_{p-q}, \quad L \geq 1, \tag{55}$$

and

$$C_p^{(L)} = \det\left\{P^{L+1} - K_0 + K_p\right\}_{(k)}. \tag{56}$$

If we multiply this determinant on the left by $\det(P^{R-L-1})(=1)$, where R is an arbitrary integer, and take the matrix P^{R-L-1} inside, (56) becomes, with the replacement $p \to k-1-q$,

$$C_{k-1-q}^{(L)} = \det\left\{P^{R} - K_0 + P^{R-L-1} J^q K_{k-1}\right\}_{(k)}. \tag{57}$$

This second form is more useful in the proof of Theorem 3 below, where we will be adding terms containing $C_p^{(L)}$'s with different L and p values ; we can sum these up into a single determinant if the base matrices are the same.

We state below some recursion relations that these coefficients satisfy:

$$\sum_{q=0}^{p} \frac{2^{2q+1} C_{p-q}^{(L+2)}}{(2q+2)!} = \sum_{q=0}^{p} \frac{C_{p-q}^{(L+1)}}{(2q+1)!} = C_p^{(L)}; \tag{58a}$$

$$\sum_{q=0}^{p} \frac{2^{2q} C_{p-q}^{(L+2)}}{(2q+1)!} = \sum_{q=0}^{p} \frac{C_{p-q}^{(L+1)}}{(2q)!} = \frac{L+1-2p}{L+1}\ C_p^{(L)}. \tag{58b}$$

(The first equality in each line also holds for $L=-1$.) Using the third equation in (55), the two equalities in (58a) follow from (13c) and (13a), respectively, and the first equality in (58b) follows from (13b). To prove the second equality, it follows from (54) that the $C_p^{(L)}$'s are the coefficients in the expansion of $(x/\sinh x)^{L+1}$:

$$\left(\frac{x}{\sinh x}\right)^{L+1} = \sum_{p=0}^{\infty} C_p^{(L)} x^{2p}. \tag{59}$$

Differentiating both sides

$$\frac{(L+1)x^L}{\sinh^{L+1} x} - \frac{(L+1)x^{L+1}\cosh x}{\sinh^{L+2} x} = \sum_{p=0}^{\infty} 2p\ C_p^{(L)} x^{2p-1}. \tag{60}$$

Rearranging terms and expanding out the $\cosh x$ factor, we get

$$\sum_{p=0}^{\infty} \frac{L+1-2p}{L+1} \; C_p^{(L)} x^{2p} = \frac{x^{L+2}\cosh(x)}{\sinh^{L+2} x} = \sum_{k=0}^{\infty}\sum_{q=0}^{\infty} C_k^{(L+1)} \; \frac{x^{2k+2q}}{(2q)!}. \tag{61}$$

The proof then follows by equating powers of x.

We define the variables x_L and y_L:

$$x_L = 2N+L+1; \quad y_L = N(N+L+1); \quad x_L^2 = 4y_L + (L+1)^2. \tag{62}$$

(To keep the notation simple, we will in general suppress the subscript L on these variables: $x_L \to x,\ y_L \to y$, with the L understood.) The generalization of Theorem 1 to hypersums is:

Theorem 2 *For $L \geq 0$ and $k > 0$, the hypersum polynomials are given by:*

$$\begin{aligned}
S_{2p}^{(2M)}(x) &= \frac{(2p)!}{2^{2p+1+2M}} \sum_{q=0}^{p+M} \left\{ C_{p+M-q}^{(2M)} \; \frac{x^{2q+1}}{(2q+1)!} - \frac{1}{(2q+1)!} \sum_{s=1}^{M} \frac{C_{p+M+1-q-s}^{(2M+2-2s)}}{(2s-1)!} \; \frac{x(x+2s-3)!!}{(x-2s+1)!!} \right\}; \\
S_{2p}^{(2M+1)}(x) &= \frac{(2p)!}{2^{2p+2+2M}} \sum_{q=0}^{p+M} \left\{ C_{p+M-q}^{(2M+1)} \; \frac{x^{2q+2}}{(2q+2)!} - \frac{1}{(2q+1)!} \sum_{s=1}^{M} \frac{C_{p+M+1-q-s}^{(2M+2-2s)}}{(2s)!} \; \frac{x(x+2s-2)!!}{(x-2s)!!} \right\}; \\
S_{2p+1}^{(2M-1)}(x) &= \frac{(2p+1)!}{2^{2p+1+2M}} \sum_{q=0}^{p+M} \left\{ C_{p+M-q}^{(2M-1)} \; \frac{x^{2q+1}}{(2q+1)!} - \frac{1}{(2q)!} \sum_{s=1}^{M} \frac{C_{p+M+1-q-s}^{(2M-2s)}}{(2s-1)!} \; \frac{(x+2s-2)!!}{(x-2s)!!} \right\}; \\
S_{2p+1}^{(2M)}(x) &= \frac{(2p+1)!}{2^{2p+2+2M}} \sum_{q=0}^{p+M} \left\{ C_{p+M-q}^{(2M)} \; \frac{x^{2q+2}-1}{(2q+2)!} - \frac{1}{(2q)!} \sum_{s=1}^{M} \frac{C_{p+M+1-q-s}^{(2M-2s)}}{(2s)!} \; \frac{(x+2s-1)!!}{(x-2s-1)!!} \right\};
\end{aligned}$$

where we use the convention that $C_p^{(L)} = 0$ for $p < 0$. See Appendix B for the proof.

The sums over q in the second terms in these expressions can be evaluated using some of the relations in (58), but the unsummed expressions above are more convenient in the proof of Theorem 3 below. Before getting to that theorem, we first need to prove a technical lemma:

Lemma 2 *The matrix P satisfies the identities,*

$$
\begin{aligned}
&\text{(I)} && D_1 P^{2k+1} D_1^{-1} J^k\, \mathbf{I}_1 = \frac{J^k}{(I-J)(I-9J)\cdots[I-(2k+1)^2J]}\, \mathbf{I}_1,\\
&\text{(II)} && D_2 P^{2k+2} D_2^{-1} J^k\, \mathbf{I}_1 = \frac{J^k}{(I-4J)(I-16J)\cdots[I-(2k+2)^2J]}\, \mathbf{I}_1,\\
&\text{(III)} && D_1 Q P^{2k+1} D_1^{-1} J^k\, \mathbf{I}_1 = \frac{J^k}{(I-4J)(I-16J)\cdots[I-(2k+2)^2J]}\, \mathbf{I}_1,\\
&\text{(IV)} && D_2 Q P^{2k+2} D_2^{-1} J^k\, \mathbf{I}_1 = \frac{J^k}{(I-J)(I-9J)\cdots[I-(2k+3)^2J]}\, \mathbf{I}_1.
\end{aligned}
$$

Proof. (We remind the reader that the D_m matrices are defined by equation (47).) Each equation above is a statement that the first column of the matrix on the left is equal to the first column of the matrix on the right. The matrices themselves are not equal since the ones on the right are LTT and those on the left are not.

Let the matrices X_k and Y_k be defined as:

$$X_k \equiv \frac{J^k}{(I-J)(I-9J)\cdots[I-(2k+1)^2J]}; \tag{63a}$$

$$Y_k \equiv \frac{J^k}{(I-4J)(I-16J)\cdots[I-(2k+2)^2J]}. \tag{63b}$$

These matrices can be expanded, respectively, in powers of odd and even integers as

$$X_k = \sum_{q=0}^{k} \frac{\alpha_q^{(k)}}{I-(2q+1)^2J} = \sum_{n=0}^{\infty} J^n \sum_{q=0}^{k} \alpha_q^{(k)} (2q+1)^{2n}, \tag{64a}$$

$$Y_k = \sum_{q=0}^{k} \frac{\beta_q^{(k)}}{I-(2q+2)^2J} = \sum_{n=0}^{\infty} J^n \sum_{q=0}^{k} \beta_q^{(k)} (2q+2)^{2n}. \tag{64b}$$

The proofs of the first equalities in (64) follow by induction on k, and then the inverse matrices are expanded in positive powers of J. It is easy to show that these sets of coefficients each satisfy $(k+1)$ independent equations,

$$\left.\begin{array}{l} \sum_{q=0}^{k} \alpha_q^{(k)} (2q+1)^{2p} \\ \\ \sum_{q=0}^{k} \beta_q^{(k)} (2q+2)^{2p} \end{array}\right\} = \left\{\begin{array}{ll} 1, & \text{for } p = k; \\ 0, & \text{for } 0 \le p < k; \end{array}\right. \tag{65}$$

(project the far left and the far right sides of equations (64) onto $\mathbf{I}_1$: in powers of J, X_k and Y_k have the form $J^k + a_1 J^{k+1} + a_2 J^{k+2} + \cdots$, so the resulting column vector's first k

elements are 0, with the $(k+1)$ elements equal to 1). The expansions for X_k and Y_k are then:

$$X_k = \sum_{n=0}^{\infty} J^{n+k} \sum_{q=0}^{k} \alpha_q^{(k)} (2q+1)^{2n+2k}; \tag{66a}$$

$$Y_k = \sum_{n=0}^{\infty} J^{n+k} \sum_{q=0}^{k} \beta_q^{(k)} (2q+2)^{2n+2k}. \tag{66b}$$

Relations (65) are sufficient to determine the $(k+1)$ values of these coefficients. The coefficients are in fact given by explicit formulas, but for the purpose of this proof, only these relations are required.

We consider relation (I) first. We will use the expansion for P in powers of J given in (41) and the $m=1$ identity in (48). For $k=0$, the left-hand side of relation I reduces to

$$D_1 P D_1^{-1}\, \mathbf{I}_1 = \sum_{q=0}^{\infty} J^q \mathbf{I}_1 = \frac{1}{I-J}\, \mathbf{I}_1, \tag{67}$$

which equals $X_0 \mathbf{I}_1$ as required. For $k>0$, an evaluation of the left-hand side of the relation results in

$$D_1 P^{2k+1} D_1^{-1} J^k\, \mathbf{I}_1 = Z_k \mathbf{I}_1 \tag{68}$$

where

$$\begin{aligned} Z_k = & \sum_{n=0}^{\infty} \frac{J^{k+n}}{(2k+1)!} \sum_{q_{2k}=0}^{n} \begin{pmatrix} 2n+2k+1 \\ 2q_{2k}+2k \end{pmatrix} \\ & \times \sum_{q_{2k-1}=0}^{q_{2k}} \begin{pmatrix} 2q_{2k}+2k \\ 2q_{2k-1}+2k-1 \end{pmatrix} \cdots \sum_{q_2=0}^{q_3} \begin{pmatrix} 2q_3+3 \\ 2q_2+2 \end{pmatrix} \sum_{q_1=0}^{q_2} \begin{pmatrix} 2q_2+2 \\ 2q_1+1 \end{pmatrix}. \end{aligned} \tag{69}$$

Z_k *is* an LTT matrix. Relation I therefore holds if and only if this matrix is equal to the matrix X_k for all k. We will assume then that, for some k, Z_k is given by the sum

$$Z_k = \sum_{n=0}^{\infty} J^{n+k} \sum_{q=0}^{k} \alpha_q^{(k)} (2q+1)^{2n+2k}. \tag{70}$$

Then

$$\begin{aligned} Z_{k+1} = & \frac{(2k+1)!}{(2k+3)!} \sum_{n=0}^{\infty} J^{k+n+1} \sum_{p_2=0}^{n} \begin{pmatrix} 2n+2k+3 \\ 2p_2+2k+2 \end{pmatrix} \\ & \times \sum_{p_1=0}^{p_2} \begin{pmatrix} 2p_2+2k+2 \\ 2p_1+2k+1 \end{pmatrix} \sum_{q=0}^{k} \alpha_q^{(k)} (2q+1)^{2p_1+2k}. \end{aligned} \tag{71}$$

Using the binomial identities,

$$\begin{aligned}
\sum_{k=0}^{n} \begin{pmatrix} 2n+2 \\ 2k+1 \end{pmatrix} x^{2k+1} &= \frac{(x+1)^{2n+2} - (x-1)^{2n+2}}{2}, \\
\sum_{k=0}^{n} \begin{pmatrix} 2n+1 \\ 2k \end{pmatrix} x^{2k} &= \frac{(x+1)^{2n+1} - (x-1)^{2n+1}}{2},
\end{aligned}$$

and relations (65) for the α's, this reduces to

$$\begin{aligned}
Z_{k+1} &= \sum_{n=0}^{\infty} J^{n+k+1} \sum_{q=0}^{k+1} (2q+1)^{2n+2k+2} \\
&\quad \times \frac{(2k+1)!}{4(2k+3)!} \left\{ \frac{2q+1}{2q-1}\, \alpha_{q-1}^{(k)} - (2+\delta_{0q})\alpha_q^{(k)} + \frac{2q+1}{2q+3}\, \alpha_{q+1}^{(k)} \right\}; \qquad (72)
\end{aligned}$$

$(\alpha_q^{(k)} = 0$ for $q < 0$ or $> k)$.

It is a straightforward calculation to show that the coefficients in the sum over q satisfy relations (65) for, and thus must be equal to, $\alpha_q^{(k+1)}$. As a consequence, $Z_{k+1} = X_{k+1}$ and, by induction on k, for all $k' > k$.

The proof of relation (II) follows by a similar argument, using the matrices D_2 and Y_k.

To prove (III), we use relation (I) to write the left-hand side of (III) as

$$D_1 Q P^{2k+1} D_1^{-1} J^k\, \mathbf{I}_1 = D_1 Q D_1^{-1} X_k\, \mathbf{I}_1. \qquad (73)$$

Evaluating this using the expansion for Q in (41b), we have:

$$\begin{aligned}
D_1 Q D_1^{-1} X_k\, \mathbf{I}_1 &= D_1 \sum_{p=0}^{\infty} \frac{J^p}{(2p)!} D_1^{-1} \sum_{m=0}^{\infty} J^{m+k} \sum_{q=0}^{k} \alpha_q^{(k)} (2q+1)^{2m+2k}\, \mathbf{I}_1 \\
&= \sum_{n=0}^{\infty} J^{n+k} \sum_{q=0}^{k} \frac{\alpha_q^{(k)}}{2(2q+1)} \left\{ (2q+2)^{2n+2k+1} + (2q)^{2n+2k+1} \right\}\, \mathbf{I}_1 \\
&= \sum_{n=0}^{\infty} J^{n+k} \sum_{q=0}^{k} (2q+2)^{2n+2k} \left\{ \frac{q+1}{2q+1}\, \alpha_q^{(k)} + \frac{q+1}{2q+3}\, \alpha_{q+1}^{(k)} \right\} \mathbf{I}_1. \qquad (74)
\end{aligned}$$

Again, it can be shown that the coefficients in this equation satisfy (65) for $\beta_q^{(k)}$, and we therefore have

$$D_1 Q P^{2k+1} D_1^{-1} J^k\, \mathbf{I}_1 = Y_k \mathbf{I}_1, \qquad (75)$$

which is relation (III).

The right-hand side of (IV) is 'almost' $X_{k+1}\mathbf{I}_1$, but it lacks one power of J. Nevertheless, we can write the left-hand side, using (II), as

$$D_2QP^{2k+2}D_2^{-1}J^k\ \mathbf{I}_1 = D_2QD_2^{-1}Y_k\ \mathbf{I}_1, \tag{76}$$

which evaluates as

$$\begin{aligned} D_2QD_2^{-1}Y_k\ \mathbf{I}_1 &= D_2\sum_{p=0}^{\infty}\frac{J^p}{(2p)!}D_2^{-1}\sum_{m=0}^{\infty}J^{m+k}\sum_{q=0}^{k}\beta_q^{(k)}(2q+2)^{2m+2k}\mathbf{I}_1 \\ &= \sum_{n=0}^{\infty}J^{n+k}\sum_{q=0}^{k}\frac{\beta_q^{(k)}}{2(2q+2)^2}\left\{(2q+3)^{2n+2k+2}+(2q+1)^{2n+2k+2}-2\right\}\mathbf{I}_1 \\ &= \sum_{n=0}^{\infty}J^{n+k}\sum_{q=0}^{k+1}(2q+1)^{2n+2k+2}\left\{\frac{\beta_q^{(k)}}{2(2q+2)^2}+\frac{\beta_{q-1}^{(k)}}{2(2q)^2}-\delta_{0q}d_k,\right\}\mathbf{I}_1 \end{aligned} \tag{77}$$

where

$$d_k \equiv \sum_{q=0}^{k}\frac{\beta_q^{(k)}}{(2q+2)^2}.$$

The coefficients in the sum over q satisfy (65) for α_q^{k+1} and therefore

$$D_2QP^{2k+2}D_2^{-1}J^k\ \mathbf{I}_1 = \sum_{n=0}^{\infty}J^{n+k}\sum_{q=0}^{k+1}\alpha_q^{(k+1)}(2q+1)^{2n+2k+2}\mathbf{I}_1, \tag{78}$$

which is relation (IV).

QED

We will use this Lemma in a slightly different form in proving Theorem 3 below. For $(k\times k)$-truncated matrices $J,\ P$, etc, the identities in the Lemma also hold if the unit column vector $\mathbf{I}_1$ is replaced by the $k\times k$ unit tower matrix K_{k-1}; both $\mathbf{I}_1$ and K_{k-1} project out the first column of any matrix that they are multiplied by from the left. We will refer to both forms (the "column-vector" and the "tower-matrix" forms) simply as "Lemma 2".

Our main result is:

Theorem 3 *Expressed in the variable $y = N(N+L+1)$, the hypersum polynomials $S_k^{(L)}(y)$ for $L\geq 0,\ k>0$ are given by*

$$\begin{aligned} S_k^{(L)}(y) &= \frac{k!}{(L+k+1)!}\ \frac{(\sqrt{4y+(L+1)^2}\,)^A}{2^{k-1}(1+L \bmod 2)}\prod_{q=0}^{\lfloor L/2\rfloor}[\ y+q(L+1-q)\ ] \\ &\times\ \det\left\{[\ I-(4y+(L+1)^2)J\ ]\prod_{q=0}^{\lfloor L/2\rfloor}[\ I-(L+1-2q)^2J\ ]\ D_mP^{L+1}D_m^{-1}-K_0+K_{n-1}\right\}_{(n)} \end{aligned}$$

where

$$
\begin{aligned}
A &= L \bmod 2 + (k+1) \bmod 2 = 0, 1, 2; \\
m &= L + 3 - k \bmod 2; \\
n &= \left\lfloor \frac{k+1}{2} \right\rfloor = p,\ p+1.
\end{aligned}
$$

Proof. We consider first the $S_{2p}^{(2M)}$ polynomials. Then $A = 1$, $m = 2M + 3$, and $n = p$. The sum over q from 0 to $p + M$ in the expression for this polynomial in Thm II requires that the size of the matrix used to represent the coefficients be at least $(p + M + 1)$. Using (57) with $R = 2M + 1$, we can express the $C_{p+M-q}^{(2M)}$ and the $C_{p+M+1-q-s}^{(2M+2-2s)}$ coefficients as determinants with the same base matrix:

$$
C_{p+M-q}^{(2M)} = \det\left\{P^{2M+1} - K_0 + J^q K_{p+M}\right\}_{(p+M+1)}; \tag{79a}
$$

$$
C_{p+M+1-q-s}^{(2M+2-2s)} = \det\left\{P^{2M+1} - K_0 + P^{2s-2} J^{q+s-1} K_{p+M}\right\}_{(p+M+1)}. \tag{79b}
$$

Then we have from Thm 2 and identity (II) in Lemma 1 that

$$
\begin{aligned}
S_{2p}^{(2M)}(x) &= \frac{(2p)!}{2^{2p+1+2M}} \sum_{q=0}^{p+M} \left\{ C_{p+M-q}^{(2M)} \frac{x^{2q+1}}{(2q+1)!} - \frac{1}{(2q+1)!} \sum_{s=1}^{M} \frac{C_{p+M+1-q-s}^{(2M+2-2s)}}{(2s-1)!} \frac{x(x+2s-3)!!}{(x-2s+1)!!} \right\} \\
&= \frac{(2p)!}{2^{2p+1+2M}} \det \left\{ P^{2M+1} - K_0 + \sum_{q=0}^{p+M} \frac{x^{2q+1}}{(2q+1)!} J^q K_{p+M} \right. \\
&\quad \left. - \sum_{q=0}^{p+M} \frac{1}{(2q+1)!} \sum_{s=1}^{M} \frac{P^{2s-2} J^{q+s-1}}{(2s-1)!} \frac{x(x+2s-3)!!}{(x-2s+1)!!} K_{p+M} \right\}_{(p+M+1)}
\end{aligned} \tag{80}
$$

This expression will be reduced and simplified by making a series of replacements for terms and factors in the tower-matrix part which are valid inside the brackets of the determinant, (but not generally outside of them). The first replacement is for the sum over q in the 4th term: $\sum_q J^q/(2q+1)! \to P$. Although P is given by an infinite sum, this replacement is allowed since P inside the brackets is truncated. Other factorials in the denominators can be dealt with by making the additional replacements:

$$
\begin{aligned}
&\sum_{q=0}^{p+M} \frac{x^{2q+1}}{(2q+1)!} J^q K_{p+M} \to D_1^{-1} \frac{x}{I - x^2 J} K_{p+M}; \\
&\frac{J^{s-1}}{(2s-1)!} K_{p+M} \to D_1^{-1} J^{s-1} K_{p+M}.
\end{aligned}
$$

Eq.(80) then becomes

$$S_{2p}^{(2M)}(x) \;=\; \frac{(2p)!}{2^{2p+1+2M}}\;\det\Bigg\{P^{2M+1}-K_0+D_1^{-1}\Bigg(\frac{x}{I-x^2J} - \sum_{s=1}^{M} D_1P^{2s-1}D_1^{-1}J^{s-1}\;\frac{x(x+2s-3)!!}{(x-2s+1)!!}\Bigg)K_{p+M}\Bigg\}_{(p+M+1)}. \tag{81}$$

The ratio of double factorials is equal to x for $s=1$ and is

$$\frac{x(x+2s-3)!!}{(x-2s+1)!!} = x(x^2-1)(x^2-9)\cdots(x^2-(2s-3)^2) \tag{82}$$

for $s>1$. We can also make the replacement

$$D_1P^{2s-1}D_1^{-1}J^{s-1} \;\rightarrow\; \frac{J^{s-1}}{(I-J)(I-9J)\cdots[I-(2s-1)^2J]}.$$

This is allowed by the "tower-matrix" form of relation (I) in Lemma 2. Upon making these changes, we get

$$S_{2p}^{(2M)}(x) \;=\; \frac{(2p)!}{2^{2p+1+2M}}\;\det\Bigg\{P^{2M+1}-K_0+D_1^{-1}\Bigg(\frac{x}{I-x^2J}-\frac{x}{I-J} - \frac{x}{I-J}\sum_{s=2}^{M}\;\prod_{q=1}^{s-1}\frac{x^2-(2q-1)^2}{I-(2q+1)^2J}\;J^{s-1}\Bigg)K_{p+M}\Bigg\}_{p+M+1} \tag{83}$$

The terms inside the brackets can be summed as

$$\frac{x}{I-x^2J}-\frac{x}{I-J}-\frac{x}{I-J}\sum_{s=2}^{M}\;\prod_{q=1}^{s-1}\frac{x^2-(2q-1)^2}{I-(2q+1)^2J}\;J^{s-1} = x\prod_{q=0}^{M-1}\frac{x^2-(2q+1)^2}{I-(2q+1)^2J}\;\frac{J^M}{I-x^2J} \tag{84}$$

by successively combining terms with the same power of J:

$$\frac{x}{I-x^2J}-\frac{x}{I-J} = \frac{x(x^2-1)}{I-J}\;\frac{J}{I-x^2J},$$

$$\frac{x(x^2-1)}{I-J}\;\frac{J}{I-x^2J}-\frac{x(x^2-1)J}{(I-J)(I-9J)} = \frac{x(x^2-1)(x^2-9)}{(I-J)(I-9J)}\;\frac{J^2}{I-x^2J},$$

etc.,

Inserting (84) into eq. (83), factoring out the $x\prod(x^2-(2q+1)^2)$ (as per identity (I) in Lemma 1), and using $J^MK_{p+M}=K_p$ (inside the brackets), we have

$$S_{2p}^{(2M)}(x) \;=\; \frac{(2p)!}{2^{2p+1+2M}}\;x\prod_{q=0}^{M-1}[\;x^2-(2q+1)^2\;] \times \det\Bigg\{P^{2M+1}-K_0+D_1^{-1}\prod_{q=0}^{M-1}\frac{1}{I-(2q+1)^2J}\;\frac{1}{I-x^2J}\;K_p\Bigg\}_{(p+M+1)}. \tag{85}$$

Now consider the value of the determinant in this equation at $x = 2M+1$. By relation (I) in Lemma 2, this is equal to

$$\det\left\{P^{2M+1} - K_0 + \frac{P^{2M+1}}{(2M+1)!}\ K_p\right\}_{(p+M+1)} . \tag{86}$$

This determinant however is zero, since it contains two columns, (the $(M+1)$-th and the last column), which are proportional to one another. We can therefore make the replacement

$$\frac{1}{I - x^2 J} \to \frac{1}{I - x^2 J} - \frac{1}{I - (2M+1)^2 J} = \frac{x^2 - (2M+1)^2}{I - (2M+1)^2 J}\ \frac{J}{I - x^2 J}$$

in the determinant in (85) without changing its value. Factoring out $x^2 - (2M+1)^2$, this equation then becomes:

$$\begin{aligned} S_{2p}^{(2M)}(x) \ &= \ \frac{(2p)!}{2^{2p+1+2M}}\ x \prod_{q=0}^{M} [\ x^2 - (2q+1)^2\ ] \\ &\times\ \det\left\{P^{2M+1} - K_0 + D_1^{-1} \prod_{q=0}^{M} \frac{1}{I - (2q+1)^2 J}\ \frac{1}{I - x^2 J}\ K_{p-1}\right\}_{(p+M+1)} . \end{aligned} \tag{87}$$

This expression accomplishes, for these values of L and k, the goal set out in the Introduction, to derive a determinant formula for the hypersum polynomials which factors out the zeros at $N = 0, \ldots, -L-1$. The size of the determinant can however be reduced; the first $(M+1)$ elements in the last column of the tower matrix in (87) are zero and we can make the replacement in that equation of

$$D_1^{-1} \to \left(\begin{array}{c|c} 0_{(M+1)} & \\ \hline & D_{2M+3}^{-1} \end{array}\right), \tag{88}$$

Successively expanding the determinant by minors along the top row, (87) is thus reduced to the $p \times p$ determinant:

$$\begin{aligned} S_{2p}^{(2M)}(x) \ &= \ \frac{(2p)!}{2^{2p+1+2M}}\ x \prod_{q=0}^{M} [\ x^2 - (2q+1)^2\ ] \\ &\times\ \det\left\{P^{2M+1} - K_0 + D_{2M+3}^{-1} \prod_{q=0}^{M} \frac{1}{I - (2q+1)^2 J}\ \frac{1}{I - x^2 J}\ K_{p-1}\right\}_{(p)} . \end{aligned} \tag{89}$$

Now applying identites (III) and (IV) in Lemma 1 and changing the variable to y using $x^2 = 4y + (L+1)^2$, eq. (89) takes the form as stated in the Theorem for $L = 2M,\ k = 2p$.

Likewise, the expressions in Theorem 2 for the other hypersums can be written in determinant form as:

$$\begin{aligned} S_{2p}^{(2M+1)}(x) &= \frac{(2p)!}{2^{2p+2+2M}} \det\Bigg\{ P^{2M+2} - K_0 + D_2^{-1}\Bigg(\frac{x^2}{I - x^2 J} \\ &- \sum_{s=1}^{M} D_2 P^{2s} D_2^{-1} J^{s-1} \frac{x(x+2s-2)!!}{(x-2s)!!} \Bigg) K_{p+M} \Bigg\}_{(p+M+1)} \end{aligned}$$

$$\begin{aligned} S_{2p+1}^{(2M-1)}(x) &= \frac{(2p+1)!}{2^{2p+1+2M}} \det\Bigg\{ P^{2M} - K_0 + D_1^{-1}\Bigg(\frac{x}{I - x^2 J} \\ &- \sum_{s=1}^{M} D_1 Q P^{2s-1} D_1^{-1} J^{s-1} \frac{(x+2s-2)!!}{(x-2s)!!} \Bigg) K_{p+M} \Bigg\}_{(p+M+1)} \end{aligned}$$

$$\begin{aligned} S_{2p+1}^{(2M)}(x) &= \frac{(2p+1)!}{2^{2p+2+2M}} \det\Bigg\{ P^{2M+1} - K_0 + D_2^{-1}\Bigg(\frac{x^2-1}{(I-J)(I - x^2 J)} \\ &- \sum_{s=1}^{M} D_2 Q P^{2s-2} D_2^{-1} J^{s-1} \frac{(x+2s-1)!!}{(x-2s-1)!!} \Bigg) K_{p+M} \Bigg\}_{(p+M+1)} \end{aligned}$$

Using relations (II), (III), and (IV), respectively, in Lemma 2, they can be brought into the forms stated in the Theorem by a similar process. For $S_{2p+1}^{(2M)}$ and $S_{2p+1}^{(2M-1)}$, the determinants corresponding to that in (86) for $S_{2p}^{(2M)}$ are not equal to zero, and so the final forms for these polynomials contain the tower matrix K_p rather than K_{p-1}. [9]
QED.

We have already given the expression for $S_6^{(10)}$ in the Introduction; some other examples are:

$$\begin{aligned} S_6^{(3)}(y) &= \frac{6!}{10!}\, y(y+3)(y+4) \times (y^2 - 2y - 1); \\ S_7^{(5)}(y) &= \frac{7!}{13!}\sqrt{y+9}\; y(y+5)(y+8) \times \frac{7y^3 + 14y^2 - 238y + 295}{7}; \\ S_{11}^{(8)}(x) &= \frac{11!}{20!}\, y(y+8)(y+14)(y+18)(y+20) \times \frac{14y^5 - 4011y^3 + 25868y^2 + 145896y - 1199616}{14} \\ S_{14}^{(14)}(y) &= \frac{14!}{29!}\, \frac{\sqrt{4y+225}}{2}\; y(y+14)(y+26)(y+36)(y+44)(y+50)(y+54)(y+56) \\ &\qquad \times (\, y^6 - 1750y^4 + 29960y^3 + 376167y^2 - 11436860y + 62455917\,) \end{aligned}$$

Now let $\Delta_k^{(L)}$ be the determinant in the expression for $S_k^{(L)}$:

$$\Delta_k^{(L)}(y) \equiv \det\left\{ [\, I - (4y + (L+1)^2)J\,] \prod_{q=0}^{\lfloor L/2 \rfloor} [\, I - (L+1-2q)^2 J\,]\; D_m P^{L+1} D_m^{-1} - K_0 + K_{n-1} \right\}_{(n)}$$

where m and n are as defined in Theorem 3.

Proposition 1 *$\Delta_k^{(L)}$ has the series expansion*

$$\Delta_k^{(L)}(y) \;=\; 4^n \sum_{s=1}^{n} y^{n-s}\, \frac{a_s}{4^s}$$

where a_s is the $s \times s$ determinant

$$a_s \;=\; \det \left\{ [I-(L+1)^2 J]^{n+1-s} \prod_{q=0}^{\lfloor L/2 \rfloor} [I-(L+1-2q)^2 J] \; D_{L+k+3-2s} P^{L+1} D_{L+k+3-2s}^{-1} - K_0 + K_{s-1} \right\}_{(s)} .$$

Proof : We re-express $\Delta_k^{(L)}$ as

$$\Delta_k^{(L)}(y) \;=\; \det \left\{ \prod_{q=0}^{\lfloor L/2 \rfloor} [\; I-(L+1-2q)^2 J\;] D_m P^{L+1} D_m^{-1} - K_0 + \frac{1}{I-[4y+(L+1)^2]J}\; K_{n-1} \right\}_{(n)} . \tag{90}$$

We have

$$\frac{1}{I-[4y+(L+1)^2]J} \;=\; \sum_{q=0}^{\infty} [4y+(L+1)^2]^q J^q = \sum_{q=0}^{\infty}\sum_{s=0}^{q} \begin{pmatrix} q \\ s \end{pmatrix} (4y)^s (L+1)^{2q-2s} J^q, \tag{91}$$

although only the $q \leq n-1$ terms in the q-sum will contribute to the determinant. We interchange the order of the sums and take the sum over s outside of the determinant:

$$\Delta_k^{(L)}(y) \;=\; \sum_{s=0}^{n-1} (4y)^s \det \left\{ \prod_{q=0}^{\lfloor L/2 \rfloor} [\; I-(L+1-2q)^2 J\;] D_m P^{L+1} D_m^{-1} - K_0 \right.$$
$$\left. + \sum_{q=s}^{n-1} \begin{pmatrix} q \\ s \end{pmatrix} (L+1)^{2q-2s} J^q K_{n-1} \right\}_{(n)} . \tag{92}$$

Using the negative binomial series,

$$\sum_{r=0}^{\infty} \begin{pmatrix} r+s \\ s \end{pmatrix} z^r = \left(\frac{1}{1-z} \right)^{s+1}, \tag{93}$$

the sum over q becomes

$$\sum_{q=s}^{n-1} \begin{pmatrix} q \\ s \end{pmatrix} (L+1)^{2q-2s} J^q K_{n-1} \;=\; \sum_{q=s}^{\infty} \begin{pmatrix} q \\ s \end{pmatrix} (L+1)^{2q-2s} J^q K_{n-1}$$
$$= \left(\frac{1}{I-(L+1)^2 J} \right)^{s+1} K_{n-1-s}. \tag{94}$$

We now make the change of summation index $s \to n - s$ and get

$$\Delta_k^{(L)}(y) = \sum_{s=1}^{n} (4y)^{n-s} a_s, \tag{95}$$

where

$$a_s = \det \left\{ [I - (L+1)^2 J]^{n+1-s} \prod_{q=0}^{\lfloor L/2 \rfloor} [I - (L+1-2q)^2 J] D_m P^{L+1} D_m^{-1} - K_0 + K_{s-1} \right\}_{(n)} .$$

As was done in the proof of Theorem 3, the size of the determinant can be reduced from $n \times n$ to $s \times s$ by the replacements $D_m^{\pm 1} \to D_{m+2n-2s}^{\pm 1}$. We recall from Theorem 3 that $m = L + 3 - k \bmod 2$ and $n = \lfloor (k+1)/2 \rfloor$; then $m + 2n - 2s$ is equal to $L + k + 3 - 2s$ and the result follows.
QED

V. CONCLUSION

We have derived a formula for the hypersum $S_k^{(L)}$ polynomial in a factored form which does not explicitly involve either Bernoulli or Stirling numbers. The hard part of the calculation has been reduced to the multiplication of matrices constructed entirely from 'simple' numbers and to the calculation of a determinant of size $\lfloor (k+1)/2 \rfloor$.

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Appendix A: Recursion relations

As noted above, recursion relation (13a) follows directly from inserting either (12a) into (11a) or (12b) into (11b). The demonstration of most of the remaining relations is more transparent (or at least more compact) if these are written in matrix form. Matrices P, Q, and J and the column vectors $\mathbf{I}_p$ were previously defined in section III; we also define the

diagonal matrix,

$$\mathit{2} \equiv \begin{pmatrix} 2^0 & & & & \\ & 2^2 & & \text{\Large 0} & \\ & & 2^4 & & \\ & \text{\Large 0} & & 2^6 & \\ & & & & \ddots \end{pmatrix}, \tag{A1}$$

and the column vectors,

$$\mathbf{C} \equiv \sum_{q=0}^{\infty} C_q \mathbf{I}_{q+1}, \quad \bar{\mathbf{C}} \equiv \sum_{q=0}^{\infty} \bar{C}_q \mathbf{I}_{q+1}, \quad \bar{C}_q \equiv \frac{E_{2q}}{(2q)!}, \tag{A2}$$

The matrix elements of P^{-1} are proportional to the Bernoulli numbers and those of Q^{-1} are proportional to the Euler numbers:

$$P^{-1} = \sum_{q=0}^{\infty} C_q J^q = \sum_{q=0}^{\infty} \frac{2-2^{2q}}{(2q)!} B_{2q} J^q, \tag{A3}$$

$$Q^{-1} = \sum_{q=0}^{\infty} \bar{C}_q J^q = \sum_{q=0}^{\infty} \frac{E_{2q}}{(2q)!} J^q. \tag{A4}$$

In this matrix notation, (13a) is

$$\mathbf{C} = P^{-1}\mathbf{I}_1. \tag{A5}$$

Now consider the product PQ:

$$\begin{aligned} PQ = \sum_{p,q=0}^{\infty} \frac{J^{p+q}}{(2p+1)!(2q)!} &= \sum_{n=0}^{\infty} J^n \sum_{q=0}^{n} \frac{1}{(2n-2q+1)!(2q)!} \\ &= \sum_{n=0}^{\infty} \frac{J^n}{(2n+1)!} \sum_{q=0}^{n} \begin{pmatrix} 2n+1 \\ 2q \end{pmatrix}. \end{aligned} \tag{A6}$$

The evaluation of the sum over q is straightforward and equals 2^{2n}, so we have

$$PQ = \sum_{n=0}^{\infty} \frac{2^{2n} J^n}{(2n+1)!}. \tag{A7}$$

In similar fashion, the products P^2 and Q^2 can be evaluated:

$$P^2 = \sum_{n=0}^{\infty} \frac{2^{2n+1} J^n}{(2n+2)!}, \tag{A8a}$$

$$Q^2 = J^0 + \sum_{n=0}^{\infty} \frac{2^{2n+1} J^{n+1}}{(2n+2)!} = I + JP^2. \tag{A8b}$$

On the other hand,

$$\mathit{2}\,J^n\,\mathit{2}^{-1} = 2^{2n}J^n, \tag{A9}$$

and therefore

$$\mathit{2}\,P\,\mathit{2}^{-1} = \sum_{n=0}^{\infty} \frac{\mathit{2}J^n\,\mathit{2}^{-1}}{(2n+1)!} = PQ \tag{A10}$$

Also,

$$JP^2 = \sum_{n=0}^{\infty} \frac{2^{2n+1}J^{n+1}}{(2n+2)!} = \sum_{p=1}^{\infty} \frac{2^{2p-1}J^p}{(2p)!} = \frac{1}{2}\left\{\sum_{p=0}^{\infty} \frac{2^{2p}J^p}{(2p)!} - I\right\}, \tag{A11}$$

or

$$\mathit{2}Q\mathit{2}^{-1} = I + 2\cdot JP^2, \quad (= 2\cdot Q^2 - I \text{ from (A8b)}). \tag{A12}$$

Now relation (13b)

$$\sum_{q=0}^{p} \frac{2^{2q}C_{p-q}}{(2q+1)!} = \frac{1}{(2p)!} \tag{A13}$$

corresponds to the matrix equation

$$\mathit{2}P\mathit{2}^{-1}\mathbf{C} = Q\mathbf{I}_1 \tag{A14}$$

and this is derived by multiplying both sides of the equation $\mathbf{C} = P^{-1}\mathbf{I}_1$ on the left by $PQ(= QP)$ and using relation (A10). The third relation, (13c),

$$\sum_{q=0}^{p} \frac{2^{2q+1}C_{p-q}}{(2q+2)!} = \frac{1}{(2p+1)!} \tag{A15}$$

is equivalent, under the substitutions $\bar{p} = p+1$ and $\bar{q} = q+1$, (and then dropping the 'bar' notation), to

$$\sum_{q=0}^{p} \frac{2^{2q}C_{p-q}}{(2q)!} = C_p + \frac{2}{(2p-1)!}.$$

In this form, (13c) is (A12) above multiplied on the right by $\mathbf{C}$:

$$\mathit{2}Q\mathit{2}^{-1}\mathbf{C} = \mathbf{C} + 2\cdot JP\mathbf{I}_1. \tag{A16}$$

The next three recursion relations

$$(13\text{d}): \quad \sum_{q=0}^{p} \frac{2^{2p-2q}C_{p-q}}{(2q)!} = C_p, \tag{A17a}$$

$$(13e): \quad \sum_{q=0}^{p} \frac{2^{2p-2q} C_{p-q}}{(2q+1)!} = \frac{E_{2p}}{(2p)!}, \tag{A17b}$$

$$(13f): \quad \sum_{q=0}^{p} \frac{E_{2q} C_{p-q}}{(2q)!} = 2^{2p} C_p, \tag{A17c}$$

all follow from relation (A10). In matrix notation, (13d) is

$$(\mathit{2}^{-1} Q \mathit{2})\mathbf{C} = \mathit{2}^{-1}\mathbf{C}, \quad \text{or} \quad Q\mathit{2}\mathbf{C} = \mathbf{C}. \tag{A18}$$

(13d) is proven by noting that the left-hand side of the equation on the right is

$$Q\mathit{2}\mathbf{C} = (P^{-1}\mathit{2}P\mathit{2}^{-1})\mathit{2}(P^{-1}\mathbf{I}_1) = P^{-1}\mathit{2}\mathbf{I}_1 = P^{-1}\mathbf{I}_1 = \mathbf{C}, \tag{A19}$$

(since $\mathit{2}\mathbf{I}_1 = \mathbf{I}_1$). Now inverting and rearranging equation (A10), we have

$$\mathit{2}P^{-1}\mathit{2}^{-1}P = Q^{-1}. \tag{A20}$$

The left-hand side of this equation is

$$\mathit{2}P^{-1}\mathit{2}^{-1}P = \sum_{p=0}^{\infty} J^p \sum_{q=0}^{p} 2^{2p-2q} \frac{C_{p-q}}{(2q+1)!}, \tag{A21}$$

while the right-hand side is given by (A4). Equating powers of J, we have

$$\sum_{q=0}^{p} \frac{2^{2p-2q} C_{p-q}}{(2q+1)!} = \bar{C}_p, \tag{A22}$$

which is relation (13e). Rearranging the factors in (A20), it becomes

$$\mathit{2}^{-1}Q^{-1}P^{-1}\mathit{2} = P^{-1}. \tag{A23}$$

From (A3) and (A4), this is relation (13f), apart from an overall factor of 2^{-2p}:

$$\sum_{q=0}^{p} 2^{-2p} \bar{C}_q C_{p-q} = C_p. \tag{A24}$$

To prove relation (13g),

$$\sum_{q=0}^{p} \frac{C_{p-q}}{(2q)!} = \frac{C_p}{2^{1-2p} - 1}, \tag{A25}$$

consider the matrix

$$\mathcal{M} \equiv 2 \cdot P\mathit{2}^{-1}P^{-1}Q - Q. \tag{A26}$$

The right-hand side can be rewritten as below:

$$
\begin{aligned}
\mathcal{M} &= 2\cdot(\mathcal{Z}^{-1}PQ)P^{-1}Q - Q, \quad (\text{using } P\mathcal{Z}^{-1} = \mathcal{Z}^{-1}QP) \\
&= 2\cdot \mathcal{Z}^{-1}Q^2 - Q \\
&= \mathcal{Z}^{-1}(\mathcal{Z}Q\mathcal{Z}^{-1} + I) - Q, \quad (\text{using } 2\cdot Q^2 = \mathcal{Z}Q\mathcal{Z}^{-1} + I) \\
&= (Q+I)\mathcal{Z}^{-1} - Q.
\end{aligned} \tag{A27}
$$

Therefore,

$$\mathcal{M}\mathbf{I}_1 = (Q+I)\mathcal{Z}^{-1}\mathbf{I}_1 - Q\mathbf{I}_1 = Q\mathbf{I}_1 + \mathbf{I}_1 - Q\mathbf{I}_1 = \mathbf{I}_1; \tag{A28}$$

or

$$\mathbf{I}_1 = (2\cdot P\mathcal{Z}^{-1}P^{-1}Q - Q)\mathbf{I}_1. \tag{A29}$$

Multiplying both sides of this equation by P^{-1}, it becomes

$$\mathbf{C} = (2\cdot\mathcal{Z}^{-1} - I)Q\mathbf{C}. \tag{A30}$$

Solving for $Q\mathbf{C}$, we get

$$Q\mathbf{C} = \frac{1}{2\cdot\mathcal{Z}^{-1} - I}\,\mathbf{C}, \tag{A31}$$

which is (13g). We now write the inverse matrix in this equation as

$$\frac{1}{2\cdot\mathcal{Z}^{-1} - I} = -I - \frac{2}{\mathcal{Z} - 2\cdot I}. \tag{A32}$$

Equation (A31) then becomes

$$\frac{1}{\mathcal{Z} - 2\cdot I}\mathbf{C} = -\frac{1}{2}(\mathbf{C} + Q\mathbf{C}). \tag{A33}$$

Recursion relations (13h),

$$\sum_{q=0}^{p} \frac{1}{(2q+1)!}\frac{C_{p-q}}{2^{2p-2q}-2} = -\frac{\delta_{0p}}{2} - \frac{1}{2(2p)!}, \tag{A34}$$

and (13i),

$$\sum_{q=0}^{p} \frac{1}{(2q)!}\frac{C_{p-q}}{2^{2p-2q}-2} = \frac{C_p}{2^{2p}-2} - \frac{1}{2(2p-1)!}, \tag{A35}$$

are obtained by multiplying this equation by P and by Q. Multiplying by P we get

$$P\frac{1}{\mathit{2}-2\cdot I}\mathbf{C}=-\frac{1}{2}(P\mathbf{C}+PQ\mathbf{C})=-\frac{1}{2}(I+Q)\mathbf{I}_1 \tag{A36}$$

which is (13h). If we multiply (A33) instead by Q, we have,

$$Q(\mathit{2}-2\cdot I)^{-1}\mathbf{C}=-\frac{1}{2}(Q\mathbf{C}+Q^2\mathbf{C}). \tag{A37}$$

From (A8b), $Q^{\mathit{2}}\mathbf{C}=(I+JP^{\mathit{2}})\mathbf{C}=\mathbf{C}+JP\mathbf{I}_1$, so

$$Q(\mathit{2}-2\cdot I)^{-1}\mathbf{C}=-\frac{1}{2}(Q\mathbf{C}+\mathbf{C}+JP\mathbf{I}_1)=(\mathit{2}-2\cdot I)^{-1}\mathbf{C}-\frac{1}{2}JP\mathbf{I}_1, \tag{A38}$$

which is (13i).

By eq. (54), $P^{-2}\mathbf{I}_1=\mathbf{C}^{(1)}$, where $\mathbf{C}^{(1)}$ is the column vector whose elements are the $C_p^{(1)}$ coefficients. Therefore, multiplying Eq.(A29) by P^{-2}, we get

$$\mathbf{C}^{(1)}=2\cdot P^{-1}\mathit{2}^{-1}P^{-1}Q\mathbf{I}_1-Q\mathbf{C}^{(1)} \tag{A39}$$

or

$$\begin{aligned} Q\mathbf{C}^{(1)} &= 2\cdot(P^{-1}\mathit{2}^{-1}Q)P^{-1}\mathbf{I}_1-\mathbf{C}^{(1)} \\ &= 2\cdot(\mathit{2}^{-1}P^{-2})\mathbf{I}_1-\mathbf{C}^{(1)} \quad \text{(using } P^{-1}\mathit{2}^{-1}Q=\mathit{2}^{-1}P^{-1} \text{ from (A20))} \\ &= (2\cdot\mathit{2}^{-1}-I)\mathbf{C}^{(1)} \end{aligned} \tag{A40}$$

which gives us

$$\sum_{q=0}^{p}\frac{C_{p-q}^{(1)}}{(2q)!}=(2^{1-2p}-1)C_p^{(1)} \tag{A41}$$

Comparing this relation to the second equality in (58b) for $L=0$, we get

$$C_p^{(1)}=\frac{2p-1}{1-2^{1-2p}}\;C_p \tag{A42}$$

which is (13j).

To prove relations (13k) and (13l), consider the polynomial

$$\sum_{q=0}^{p}\frac{C_{p-q}}{(2q+1)!}\;u^{2q+1}. \tag{A43}$$

Using relation (16), this is expressible in terms of Bernoulli polynomials,

$$\sum_{q=0}^{p}\frac{C_{p-q}}{(2q+1)!}\;u^{2q+1}=\frac{2}{(2p+1)!}\left[B_{2p+1}(u)-2^{2p}B_{2p+1}\left(\frac{u}{2}\right)\right]. \tag{A44}$$

As a consequence of the identity $B_k(u+1) = B_k(u) + ku^{k-1}$, the RHS becomes

$$\begin{aligned}
\sum_{q=0}^{p} \frac{C_{p-q}}{(2q+1)!}\ u^{2q+1} &= \frac{2}{(2p)!}\ (u-1)^{2p} + \sum_{q=0}^{p} \frac{C_{p-q}}{(2q+1)!}\ (u-2)^{2q+1} \\
&= \frac{2}{(2p)!} \left\{ (u-1)^{2p} + (u-3)^{2p} + \cdots + (u-2N+1)^{2p} \right\} \\
&\qquad + \sum_{q=0}^{p} \frac{C_{p-q}}{(2q+1)!}\ (u-2N)^{2q+1}; \qquad \text{(A45)}
\end{aligned}$$

Differentiating this equation with respect to u,

$$\begin{aligned}
\sum_{q=0}^{p} \frac{C_{p-q}}{(2q)!}\ u^{2q} &= \frac{2}{(2p-1)!}\ (u-1)^{2p-1} + \sum_{q=0}^{p} \frac{C_{p-q}}{(2q)!}\ (u-2)^{2q} \\
&= \frac{2}{(2p-1)!} \left\{ (u-1)^{2p-1} + (u-3)^{2p-1} + \cdots + (u-2N+1)^{2p-1} \right\} \\
&\qquad + \sum_{q=0}^{p} \frac{C_{p-q}}{(2q)!}\ (u-2N)^{2q}. \qquad \text{(A46)}
\end{aligned}$$

Setting $u = 2N$ and $u = 2N+1$ in these two sets of equations prove, respectively, relation (13k) and, using (13g), relation (13l).

Appendix B: Proof of Theorem 2

The proof is by induction on L and uses the expressions for the integer power sums from Theorem 1,

$$\begin{aligned}
S_{2p}(N) &= \frac{(2p)!}{2^{2p+1}} \sum_{q=0}^{p} C_{p-q} \frac{(2N+1)^{2q+1} - 1}{(2q+1)!} \\
&= \frac{(2p)!}{2^{2p+1}} \sum_{q=0}^{p} C_{p-q} \frac{(2N+1)^{2q+1}}{(2q+1)!} \quad (\text{for } p > 0), \qquad \text{(B1a)} \\
S_{2p+1}(N) &= \frac{(2p+1)!}{2^{2p+2}} \sum_{q=0}^{p} C_{p-q} \frac{(2N+1)^{2q+2} - 1}{(2q+2)!}, \qquad \text{(B1b)}
\end{aligned}$$

and the odd-integer power sums from Corollary 1,

$$\begin{aligned}
\bar{S}_{2p}(N) &= \frac{(2p)!}{2} \sum_{q=0}^{p} C_{p-q}\ \frac{(N+1)^{2q+1}}{(2q+1)!}, \qquad \text{(B1c)} \\
\bar{S}_{2p+1}(N) &= \frac{(2p+1)!}{2} \sum_{q=0}^{p} C_{p-q}\ \frac{(N+1)^{2q+2}}{(2q+2)!}, \qquad \text{(B1d)}
\end{aligned}$$

where in the last two equations N is an odd integer. The expressions for $S_k^{(L)}$ in Theorem 2 reduce to the integer power sums above when $L=0$. It is necessary then to show that these expressions satisfy

$$\sum_{n=1}^{N} S_k^{(L)}(n) = S_k^{(L+1)}(N). \tag{B2}$$

Since $S_k^{(L)}(n)=0$ for $n=0,-1,-2,\ldots,-L-1$, we can make the replacement

$$\sum_{n=1}^{N} S_k^{(2M,2M+1)}(n) \quad\rightarrow\quad \sum_{n=-M}^{N} S_k^{(2M,2M+1)}(n) \tag{B3}$$

without affecting the values of the sums. Taking the sum from $n=-M$ to $n=N$ of the first term in the expression for $S_{2p}^{(2M)}$ in Theorem 2, we have

$$\begin{aligned}
\sum_{n=-M}^{N}\sum_{q=0}^{p+M} C_{p+M-q}^{(2M)} \left.\frac{x^{2q+1}}{(2q+1)!}\right|_{x=2n+1+2M} &= \sum_{q=0}^{p+M} \frac{C_{p+M-q}^{(2M)}}{(2q+1)!}\,\bar{S}_{2q+1}(2N+1+2M)\\
&= \frac{1}{2}\sum_{q=0}^{p+M}\sum_{s=0}^{q} C_{p+M-q}^{(2M)} C_{q-s} \left.\frac{X^{2s+2}}{(2s+2)!}\right|_{X=2N+2+2M}\\
&= \frac{1}{2}\sum_{s=0}^{p+M}\frac{X^{2s+2}}{(2s+2)!}\sum_{q=s}^{p+M} C_{p+M-q}^{(2M)} C_{q-s}\Big|_{X=2N+2+2M}\\
&= \frac{1}{2}\sum_{s=0}^{p+M} C_{p+M-s}^{(2M+1)} \left.\frac{X^{2s+2}}{(2s+2)!}\right|_{X=2N+2+2M}
\end{aligned} \tag{B4a}$$

In the 2nd line we've used (B1d) for the sum of odd integers raised to an odd power. By similar calculations, we get for the other sums,

$$\sum_{n=-M}^{N}\sum_{q=0}^{p+M} C_{p+M-q}^{(2M+1)} \left.\frac{x^{2q+2}}{(2q+2)!}\right|_{x=2n+2+2M} = \frac{1}{2}\sum_{s=0}^{p+M+1} C_{p+M+1-s}^{(2M+2)} \left.\frac{X^{2s+1}-X}{(2s+1)!}\right|_{X=2N+3+2M} \tag{B4b}$$

$$\sum_{n=-M+1}^{N}\sum_{q=0}^{p+M} C_{p+M-q}^{(2M-1)} \left.\frac{x^{2q+1}}{(2q+1)!}\right|_{x=2n+2M} = \frac{1}{2}\sum_{s=0}^{p+M} C_{p+M-s}^{(2M)} \left.\frac{X^{2s+2}-1}{(2s+2)!}\right|_{X=2N+1+2M} \tag{B4c}$$

$$\sum_{n=-M}^{N}\sum_{q=0}^{p+M} C_{p+M-q}^{(2M)} \left.\frac{x^{2q+2}-1}{(2q+2)!}\right|_{x=2n+1+2M} = \frac{1}{2}\sum_{s=0}^{p+M+1} C_{p+M+1-s}^{(2M+1)} \left.\frac{X^{2s+1}-2^{2s}X}{(2s+1)!}\right|_{X=2N+2+2M} \tag{B4d}$$

We now consider the 2nd terms in the expressions for $S_k^{(L)}$. We can use the identities

$$\left.\frac{(x+2s-1)!!}{(x-2s-1)!!}\right|_{x=2n+1+2M} = 2^{2s}(2s)!\begin{pmatrix} n+M+s\\ 2s\end{pmatrix},$$

$$\left.\frac{(x+2s-2)!!}{(x-2s)!!}\right|_{x=2n+2+2M} = 2^{2s-1}(2s-1)!\begin{pmatrix} n+M+s\\ 2s-1\end{pmatrix},$$

$$\sum_{j=s}^{N}\begin{pmatrix} j \\ s \end{pmatrix} = \begin{pmatrix} N+1 \\ s+1 \end{pmatrix}$$

and

$$\begin{pmatrix} n+M+q+a \\ 2q+a \end{pmatrix} = \frac{2q+1+a}{2n+2M+1+a}\left\{2\begin{pmatrix} n+M+q+1+a \\ 2q+1+a \end{pmatrix} - \begin{pmatrix} n+M+q+a \\ 2q+a \end{pmatrix}\right\},$$

to derive the relations:

$$\sum_{n=-M}^{N} \left.\frac{x(x+2s-3)!!}{(x-2s+1)!!}\right|_{x=2n+1+2M} = \frac{1}{2(2s)} \left.\frac{X(X+2s-2)!!}{(X-2s)!!}\right|_{X=2N+2+2M}; \tag{B5a}$$

$$\sum_{n=-M}^{N} \left.\frac{x(x+2s-2)!!}{(x-2s)!!}\right|_{x=2n+2+2M} = \frac{1}{2(2s+1)} \left.\frac{X(X+2s-1)!!}{(X-2s-1)!!}\right|_{X=2N+3+2M}; \tag{B5b}$$

$$\sum_{n=-M+1}^{N} \left.\frac{(x+2s-2)!!}{(x-2s)!!}\right|_{x=2n+2M} = \frac{1}{2(2s)} \left.\frac{(X+2s-1)!!}{(X-2s-1)!!}\right|_{X=2N+1+2M}; \tag{B5c}$$

$$\sum_{n=-M}^{N} \left.\frac{(x+2s-1)!!}{(x-2s-1)!!}\right|_{x=2n+1+2M} = \frac{1}{2(2s+1)} \left.\frac{(X+2s)!!}{(X-2s-2)!!}\right|_{X=2N+2+2M}. \tag{B5d}$$

With the sets of relations (B4) and (B6) it is straightforward to show that the expressions in Theorem 2 for the hypersums satisfy condition (B2) above. For example, with $x \equiv 2n+1+2M$ and $X \equiv 2N+2+2M$,

$$\begin{aligned}
\sum_{n=1}^{N} S_{2p+1}^{(2M)}(n) &= \sum_{n=-M}^{N} S_{2p+1}^{(2M)}(n) \\
&= \frac{(2p+1)!}{2^{2p+2+2M}} \sum_{n=-M}^{N}\sum_{q=0}^{p+M}\left\{C_{p+M-q}^{(2M)}\frac{x^{2q+2}-1}{(2q+2)!} - \frac{1}{(2q)!}\sum_{s=1}^{M}\frac{C_{p+M+1-q-s}^{(2M-2s)}}{(2s)!}\frac{(x+2s-1)!!}{(x-2s-1)!!}\right\} \\
&= \frac{(2p+1)!}{2^{2p+3+2M}} \sum_{q=0}^{p+M+1}\left\{C_{p+M+1-q}^{(2M+1)}\frac{X^{2q+1}-2^{2q}X}{(2q+1)!} - \frac{1}{(2q)!}\sum_{s=1}^{M}\frac{C_{p+M+1-q-s}^{(2M-2s)}}{(2s+1)!}\frac{(X+2s)!!}{(X-2s-2)!!}\right\} \\
&= \frac{(2p+1)!}{2^{2p+3+2M}} \sum_{q=0}^{p+M+1}\left\{C_{p+M+1-q}^{(2M+1)}\frac{X^{2q+1}}{(2q+1)!} - \frac{1}{(2q)!}\sum_{s=0}^{M}\frac{C_{p+M+1-q-s}^{(2M-2s)}}{(2s+1)!}\frac{(X+2s)!!}{(X-2s-2)!!}\right\} \\
&= \frac{(2p+1)!}{2^{2p+3+2M}} \sum_{q=0}^{p+M+1}\left\{C_{p+M+1-q}^{(2M+1)}\frac{X^{2q+1}}{(2q+1)!} - \frac{1}{(2q)!}\sum_{s=1}^{M+1}\frac{C_{p+M+2-q-s}^{(2M+2-2s)}}{(2s-1)!}\frac{(X+2s-2)!!}{(X-2s)!!}\right\} \\
&= S_{2p+1}^{(2M+1)}(N),
\end{aligned} \tag{B6}$$

where we've used the relation

$$\sum_{q=0}^{p}\frac{2^{2q}C_{p-q}^{(L+2)}}{(2q+1)!} = \sum_{q=0}^{p}\frac{C_{p-q}^{(L+1)}}{(2q)!} \tag{B7}$$

from relations (58b) to move the $2^{2q}X$ term in the first summation to the second summation where it becomes the $s = 0$ term, and then relabeled the index, $s \to s - 1$.
QED

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[9] Except for the $L = 0$, $k = 2p + 1 > 1$ case, which we've seen can be reduced to a $p \times p$ determinant, a result of the identity $(I - J)^2 D_2 P D_2^{-1} \mathbf{I}_1 = \mathbf{I}_1$.

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